An Algorithm for Adjustment of Geometric Levelling Networks

Vasil Cvetkov [1*)]

[1*)] University of Architecture, Civil Engineering and Geodesy, Geodetic Department, 1 Hristo Smirnenski Blvd., 1164 Sofia, Bulgaria; email: tzvetkov_fgs@uacg.bg; https://orcid.org/0000-0001-9628-6768

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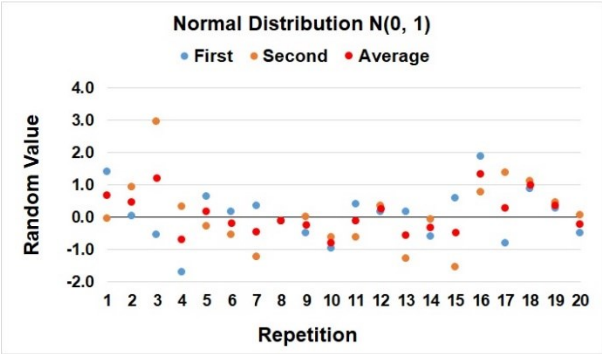
Abstract

The natural way to reduce the duration of measurement of a levelling network is to cut down on the number of levelling lines without damaging the quality of the final results. The main objective of the study is to demonstrate that this is possible without any lack of accuracy, if some mathematical facts regarding the average of both measurements of the line elevations are taken into account. Based on 60 paired random samples of size 1000, derived from different continuous distributions, e.g., N (0, 1), U (-1.732, 1.732) and Gamma (1, 1), each of them with theoretical standard deviation σ=1, it was found that the averages of each pair form new distribution with standard deviation σ≈0.707. However, the samples, which were formed by selecting the nearest to the known theoretical expectation from both measurements and their average have distributions, which standard deviations tend to σ≈0.53, σ≈0.46 and σ≈0.43 for the U (-1.732, 1.732), N (0, 1) and Gamma (1, 1) distributions, respectively. Therefore, if we choose the more appropriate value from the "first", the "second" measurement and their average, we will increase the accuracy of the network almost $\sqrt{2}$ times in comparison to the accuracy, yielded by the only use of the averages. If our network contains n lines, the process of finding of these elevation values, which leads to the best fit of the network, is based on 3^n single adjustments of the network. In addition, we can minimize the impact of the shape of the network on the final standard errors of the adjusted heights or geopotential numbers of the nodal benchmarks in the network, if we apply some iterative procedures, e.g., Inverse Distance Weighting (IDW), Inverse Absolute Height Weighting (IAHW), etc. In order to check the above explained algorithm, the Second Levelling of Finland network was adjusted in three variants. In the first variant, the whole network was adjusted as a free one. The classical weights $w=L^{-1}$ were used. In the second variant, the network was separated into two parts. Applying 3^{12} and 3^{14} independent adjustments, the selection of the best fitted values of line elevations was done and the network was adjusted by using them. The IDW and IAHW with power parameter p=5 were finally applied. In the third variant, the network was separated in four parts. Applying 3^{13}, 3^{12}, 3^{16} and 3^{12} independent adjustments, the new selection of the line elevations was done and the network was adjusted by them. The IDW (p=6.5) and IAHW (p=6) were executed. Comparison of the standard errors of the adjusted geopotential numbers in the separate variants revealed that there was no statistically significant difference between the results, yielded in the second and the third variant. However, these variants produced 3-5 times increase of the accuracy in comparison to the classical first variant. The best results were obtained in the second variant with IAHW, where the mean value of the standard errors of the adjusted geopotential numbers is below 1.4 mgpu.

Keywords: algorithm, adjustment, geometric levelling networks, accuracy

Introduction

The highest order geometric levelling has been used as a main method for establishing of height system for state territories [1, 2] since the last decades of the 19th century. The method is also widely applied for verification of monitoring of the recent vertical movements of the Earth's crust [3, 4], verification and calibration of GNSS measurement [5] and chronometric levelling results [6]. The importance of the precise geometric levelling for civil engineering activities is also well-known [7]. However, geometric levelling has its downsides. One of them is that the method is time-consuming. The duration of measurement of a state levelling network is usually years, but in some cases and decades [1, 8]. During the campaign time there are many natural processes, which lead to deterioration of levelling networks and their accuracy, e.g., the recent vertical movements of the Earth's crust [3], tidal effects [6], etc. Therefore, the fast measurement of a levelling network has (a) reflection on the accuracy of the network. A natural way to reduce the duration of measurement of a network is to cut down on the number of levelling lines without damaging the quality of the final results. Is it possible?



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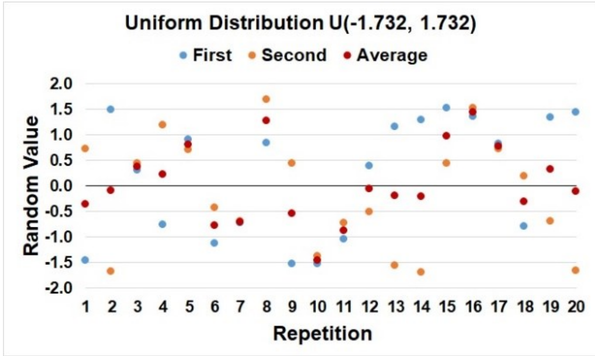

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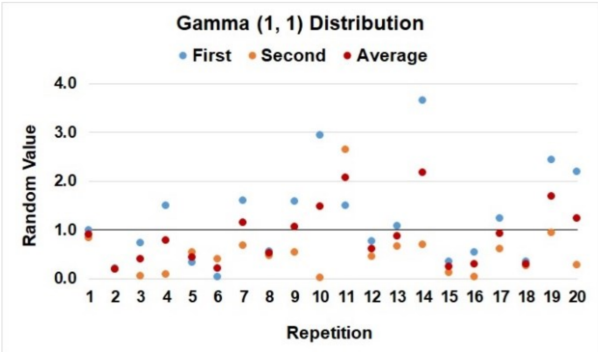

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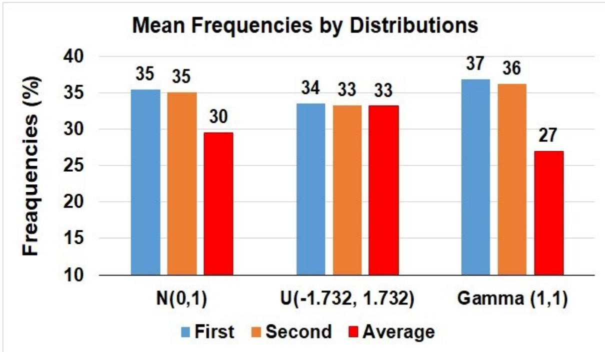
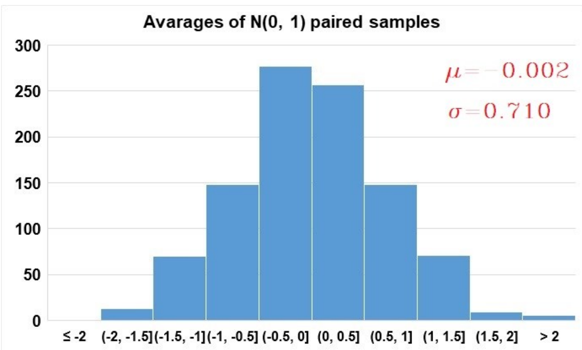


(d)

Fig. 1. Random location of the "first", the "second" observation or their mean regarding the distribution theoretical expectation: a) N(0, 1) with expectation μ=0; b) U(-1.732, 1.732) with expectation μ=0; c) Gamma (1, 1) with expectation μ=1; d) Mean frequencies to be some observation nearby the theoretical expectation, based on 60 independent paired random samples of size 1000.

As a common rule, the elevation between each benchmark in a levelling line is measured twice. Both measurements are in opposite directions. Thus, we have an upward elevation and a downward elevation. The idea of the opposite measurements is the effect of some systematic errors to be reduced in the average elevation. Also, each of both measurements "guarantee" that in another measurements there is no gross errors. If the range, that is to say the difference between both measurements, is an "acceptable", we form the average of two observations, supposing that this average tends to the true value of the measured elevation. Simulations of double measurements, illustrated by Figure 1, reveal another different picture. According to Figure 1, the average of two same distributed and ordered observations is very nearby to the theoretical expectation, in comparison to both observations, only in approximately 27-30% of all cases. Contrary, in other 70-74% of cases, either the "first" or the "second" observation is in close proximity to the expectation. What is more important, is that the distribution of the averages has standard deviation $\sigma/\sqrt{2}$, if σ is the standard deviation of the distribution of the "first" and the "second" observation. This fact is clearly shown by the histograms in Figure 2. These histograms reveal additional fact. The samples, which were formed by selecting the nearest to the known theoretical expectation from both measurements and their average have distributions, whose standard deviations tend to σ≈0.53, σ≈0.46 and σ≈0.43 for the U (-1.732, 1.732), N (0, 1) and Gamma (1, 1) distributions, respectively, when the sample size tend to infinity. Therefore, if we choose the more appropriate value from the "first", the "second" measurement and their average, we will increase the accuracy of the network almost $\sqrt{2}$ times in comparison to the accuracy, yielded by the only use of the averages.



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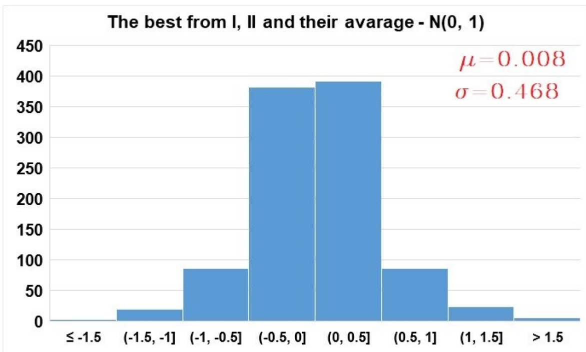

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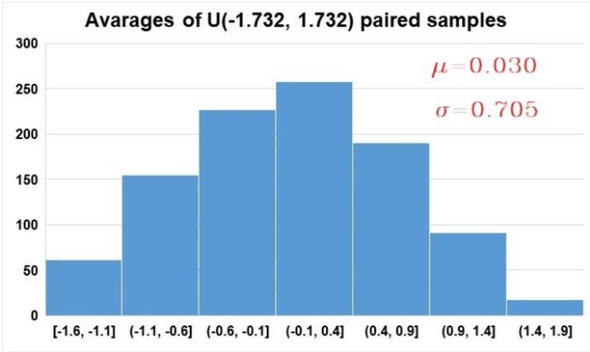

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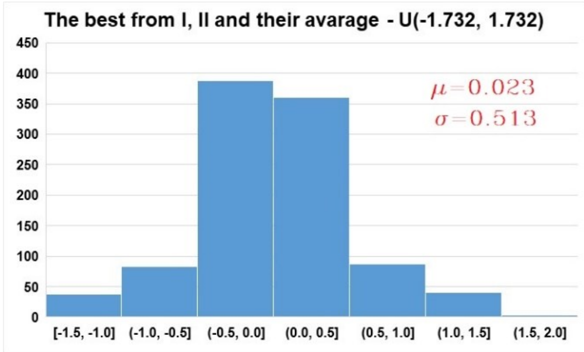

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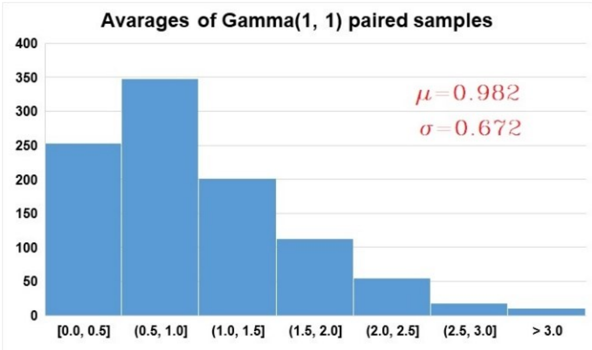

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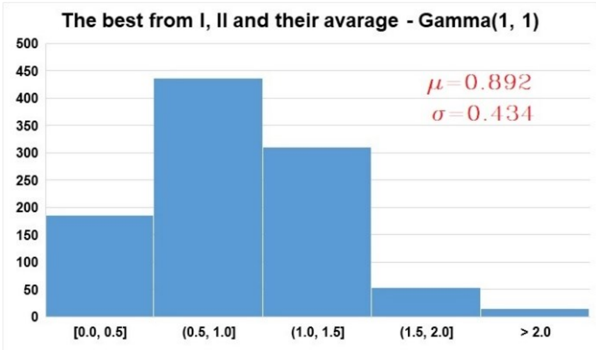

(f)

Fig. 2. Histograms: (left) of the means of the "first" and the "second" observation, based on independent paired random samples of size 1000; (right) of the best from I, II and their mean; a) and b) - N(0, 1); c) and d) - U(-1.732, 1.732); e) and f) - Gamma (1, 1)

If our network contains n lines, the process of finding of these elevation values, which leads to the best fit of the network, is based on 3^n single adjustments of the network. If the treated network is too big similar adjustments may not be possible with nowadays computers, but it will be possible in the coming years. For now we can only use "hungry" algorithms of the type "divide et impera" to simplify the computation task.

In addition, we can minimize the impact of the shape of the network on the final standard errors of the adjusted heights or geopotential numbers of the nodal benchmarks in the network, if we apply some iterative procedures, e.g., Inverse Distance Weighting (IDW), Inverse Absolute Height Weighting (IAHW) [10], etc.

The main objective of the study is to present an algorithm of a modern adjustment of geometric levelling networks, which allows us to reduce the number of levelling lines in a network without any lack of accuracy. The last is possible, if some mathematical facts regarding the average of both measurements of the line elevations and the minimization of the standard errors of the adjusted geopotential numbers are taken into account. Last but not least, another purpose of the text is to demonstrate that the accuracy of the highest order geometric levelling is supreme, at the moment, in comparison to other present methods for determining of vertical differences between points on the Earth's surface.

Methods

In order to check the above statements, the Second Levelling of Finland network, was adjusted in three variants.

In Variant 1, the original configuration of the network, which can be found in the study by Kääriäinen [8, p. 11], was kept. The weights, used in the adjustment are given by equation (1), where the power parameter p=1. Therefore, the classic weights applied to the adjustment of geometric levelling networks, were used. In equation (1), n denotes the number of levelling lines which form the network.

$$w_n = \left(\frac{const.}{L_n}\right)^p \tag{1}$$

In Variant 2, the adjustment of the network was performed in five steps.

1. The network was separated into two parts, the southern and the northern ones. The common lines between both parts were lines 24, 26, 27 and 29. Standard errors of the adjusted geopotential numbers of the nodal benchmarks, produced by Variant 1, are given by table 2 /column 2/ in the Results section.
2. The southern network was organized in four loops. The first and the second loops combined the original loops I, II and VI, VII, respectively. Lines 2 and 20 were not used in the adjustment. The third loop was formed by the original loops III and VIII. Thus, line 9 was skipped. The fourth loop was the original loop IX. In order to find those values of the "first", the "second" and the "mean", which minimize the mean error per weight unit, the network was adjusted 3^{12} times. Weights (1) and power parameter p=1 in these adjustments were used.
3. The northern part was reconstructed in five loops. The first and the second were the original loops X and XI, respectively. The third loop was formed by the original loops XII and XV. The fourth loop contained loops XIII and XIV. The fifth one combined loops XVI and XVII. Thus, lines 39, 42, 51, 59, 60, 61, 62 and 63 were not included in the adjustment. The

northern network was adjusted 3^{14} independent times in search of these values of line elevations, which best fit it. Weights (1) and power parameter p=1 in these adjustments were used.

4. Finally, the network of Variant 2 was constructed by these values of the line geopotential number differences, which fitted the southern and the northern parts in the best way. In case of the common lines, the mean of the "best fitting" values were taken in the adjustment.
5. In order to minimize the impact of network configuration, plain and vertical, the final network was adjusted by iterative procedures, e.g., Inverse Distance Weighting (IDW) and Inverse Absolute Height Weighting (IAHW), applying different values of the power parameter p in weights (1) and (2). Standard errors of the adjusted geopotential numbers of the nodal benchmarks, produced by Variant 2, are shown by table 1 in the "Results" section.

$$w_n = \left(\frac{const.}{|H|_n}\right)^p \quad (2)$$

In Variant 3, the original network was separated into four parts. The first one included loops I, II, VI and VII. The second part was formed by loops III, VIII and IX. The third one combined loops X, XI, XII, XIII and XIV. The fourth was constructed by loops XV, XVI, XVII and XVIII. Each part was adjusted in full combination. Therefore, 3^{13}, 3^{12}, 3^{16} and 3^{12} independent adjustments were performed in searching for these values, which fit the parts in the best way. Using selected values of the line geopotential number differences, the whole network was adjusted by IDW and IAHW with different values of the power parameter p. The results, produced by this variant are given by table 2.

All final adjustments were performed by the parametric approach. The nodal bench mark in Qulunkyla was used as a datum point.

Results

The standard errors of the adjusted geopotential numbers of the nodal benchmark in the network of Variant 2, obtained by the adjustment of the network by the classical approach and by the methods of Variant 2, are given by table 1.

Tab. 1. Standard errors of the adjusted geopotential numbers of the nodal benchmarks, produced by Variant 2.

Benchmark	Classic SE (mgpu)	BF[a] SE (mgpu)	BF[a]+ IDW(5) SE (mgpu)	BF[a] + IAHW(5) SE (mgpu)	Distance (km)
Toijala	6.24	3.36	1.43	1.29	146.19
Kouvola	7.67	4.13	2.24	1.50	185.03
Turku	6.74	3.63	1.61	0.00	208.29
Haapamaki	7.82	4.21	2.06	1.27	307.92
Pieksamaki	8.46	4.56	2.08	1.55	370.64
Jyvaskyla	8.40	4.53	2.07	0.33	389.01
Sarkisalmi	9.49	5.11	2.70	1.50	391.38
Noormarkku	8.70	4.68	2.70	0.17	417.22
Seinajoki	9.12	4.91	2.23	0.29	423.26
Kontiomaki	10.61	5.71	4.56	2.04	657.24
Ylivieska	10.16	5.47	4.17	0.29	701.80
Oulu E.	10.89	5.87	4.25	2.39	822.66
Rovaniemi	13.26	7.14	6.38	4.42	1048.51

[a] Best Fitting, yielded by 3^n independent adjustments.

The standard errors of the adjusted geopotential numbers of the nodal benchmark in the whole network, obtained by the adjustment of the network by both Variant 1 and Variant 3, are given by table 2. Table 1 and table 2 also contain the remoteness of each nodal benchmark from the chosen datum point in Qulunkyla.

Comparing the standard errors of the adjusted geopotential numbers of the common benchmarks, given in table 1 and table 2, it is possible to reduce the number of levelling lines in a network without any lack of accuracy, if some mathematical facts regarding the average of both measurements of the line elevations and the minimization of the standard errors of the adjusted geopotential numbers are taken into account. The values of the standard errors, included in the second and the third columns of table 1 and table 2, prove the theory regarding the means of two measurements and their accuracy, explained in the Introduction section. Additionally, iterative procedures as the IDW and IAHW, increase the accuracy of levelling networks. According to the obtained results, the proposed procedures give opportunity to yield standard errors of the adjusted geopotential numbers with the median value below 1.5 mgpu.

Tab. 2. Standard errors of the adjusted geopotential numbers of the nodal benchmarks, produced by Variant 1 and Variant 3.

Benchmark	Classic SE (mgpu)	BF[a] SE (mgpu)	BF[a] + IDW(6.5) SE (mgpu)	BF[a] + IAHW(6) SE (mgpu)	Distance (km)
Karjaa	4.40	2.32	0.55	0.01	88.99
Hyvinkaa	3.69	1.94	0.12	1.22	98.29
Riihimaki	4.09	2.15	0.12	1.22	111.38
Toijala	5.69	3.00	0.52	1.22	146.19
Kouvola	5.53	2.91	1.73	1.21	185.03
Turku	6.09	3.21	1.25	0.02	208.29
Tampere	6.31	3.33	0.53	1.22	234.86
Haapamaki	7.10	3.74	1.77	1.28	307.92
Peipohja	6.96	3.67	1.13	0.17	362.18
Pieksamaki	7.23	3.81	1.88	1.40	370.64
Javaskyla	7.48	3.94	1.83	1.56	389.01
Sarkisalmi	7.94	4.19	4.03	1.21	391.38
Noormarkku	7.48	3.94	1.14	0.17	417.22
Seinajoki	8.21	4.32	2.38	0.22	423.26
Toivala	8.23	4.34	2.17	1.21	473.22
Iisalmi	8.56	4.51	2.20	1.21	548.47
Joensuu	8.80	4.64	3.25	1.21	556.23
Kontiomaki	9.26	4.88	2.56	5.85	657.24
Parkkirma	8.62	4.54	2.23	3.39	666.80
Ylivieska	8.89	4.69	2.29	0.22	683.95
Oulu E.	9.76	5.14	2.89	2.35	804.81
Oulu P.	9.84	5.18	2.89	2.35	810.40
Kuusamo	11.01	5.80	10.37	10.78	909.94
Laurila	11.27	5.94	3.26	2.35	923.27
Rovaniemi	11.71	6.17	3.49	3.20	1030.66

[a] Best Fitting, yielded by 3^n independent adjustments.

Discussion

According to the results, given by table 1 and table 2, the classic approach of the adjustment of geometric levelling networks is not the best one. There are other procedures which lead to minimization of the standard error of the adjusted geopotential number of nodal benchmarks in a geometric levelling network. Additionally, the alternative methods minimize the effect of the remoteness of a nodal bench mark from the datum point in the network. These weak points of the classic method will be discussed in the text below.

1. Comparison between the standard error of geopotential numbers of the nodal benchmarks in the analyzed networks, given in the second and the third columns in table 1 and table 2, shows that applying an adjustment in all combinations with the use of the “first”, the “second” observations and their means, leads to almost twice less standard errors than the classic adjustment with usage only of the means. This fact proves the theory which was explained in the Introduction rather than be a surprise. The Paired Two-Sample for Means t-Tests using the above mentioned data, reject the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ at 99.99% confidence level in both cases. The actual p-values concerning the data in table 1 and table 2 are $7.4.10^{-10}$ and $2.5.10^{-14}$. Consequently, the 3^n independent adjustments procedure leads to statistically significant improvement of the adjustment results.
2. Additional minimizing of the adjusted standard errors of the nodal benchmarks in the analyzed network was done possible by applying the IDW and IAHW procedures. The Paired Two-Sample for Means t-Tests based on the data in columns 3 and 4 in table 1 and table 2, reject the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ at 99.99% confidence level in both cases. The actual p-values concerning the data in table 1 and table 2 are $4.5.10^{-8}$ and $3.6.10^{-6}$. Therefore, the 3^n independent adjustments procedure plus some IDW or IAHW procedures lead to statistically significant better results than only 3^n independent adjustments variant.
3. According to the data, given in table 1 and table 2, the best results were obtained in the 3^n independent adjustments procedure plus some IAHW variants. The Paired Two-Sample for Means t-Test based on the data in columns 4 and 5 in table 1, rejects the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ at 99.99% confidence level. The actual p-value is $6.9.10^{-6}$. Similar statistical significance regarding the data in table 2 was not found. Thus, in order to get the best results, the performing of both procedures is preferable.
4. Comparison of the data in the second columns in table 1 and table 2 supports the theory, which states, the more observations are made, the higher accuracy will be obtained. The Paired Two-Sample for Means t-Test, rejects the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$. The obtained p-value is $4.8.10^{-7}$. (A) Similar situation we have regarding the data in the fourth columns in both tables, where the standard errors obtained from the 3^n independent adjustments plus IDW procedures were performed. The p-value is 0.019. Thus, we can reject the null hypothesis $H_0 : \mu_1 = \mu_2$ against the alternative hypothesis $H_1 : \mu_1 \neq \mu_2$ at 98% level. However, the best results, which are given in the fifth column of the tables above, reveal another picture. The Paired Two-Sample for Means t-Test based on these data supports the null hypothesis $H_0 : \mu_1 = \mu_2$. The obtained p-value in two-sided test is 0.52. Accordingly, we might state that the more observations were made,

the higher accuracy would not be obtained. The last statement is fully supported by the results of the Third Levelling of Finland [11], where including some new lines in order to thicken the Second Levelling network did not increase the accuracy of the network. For more details, one can see Figure 6.3 in [11].

5. A huge weak point of the classical approach of the adjustment of the highest order levelling networks is a strong correlation between the standard errors of the adjusted geopotential numbers / heights of the nodal benchmarks and their remoteness from the datum point [6, 11-12]. Analyzing the data included in the second and the sixth columns in table 1 and table 2, one can calculate that the correlation coefficients between the standard errors of the adjusted heights of the benchmarks and their remoteness from the datum point in Qulunkyla are 0.96 and 0.98, respectively for the data in table 1 and table 2. Such a strong correlation leads to systematic increase of the standard errors of the adjusted heights of benchmarks in respect of remoteness. Regarding the Russian First Order Levelling Network [12], this fault of the classical adjustment produced systematic increase of the standard errors of the benchmarks with remoteness from the peil in Kronstadt. The standard error of the height of the most remoted benchmark in Amguema (on Bering see) is estimated to be 148.9 mm. The same behavior have the standard errors pictured by Figure 6.3 in [11]. However, analyzing the data in columns 5 and 6 of table 1 and table 2, it is easy to be calculated that the correlation coefficients between the standard errors of the adjusted geopotential numbers of the benchmarks and their remoteness from the datum point in Qulunkyla are 0.65 and 0.57, respectively for the data in table 1 and table 2. Based on the magnitude of the standard errors in columns 1 and 5, one may speculate that if the Russian First Order Levelling Network [12] was adjusted by the 3^n independent adjustments plus IAHW procedure, the standard error of the height of the benchmark in Amguema might be estimated to be below 30 mm. This matter needs further investigations.

Conclusion

In this paper was demonstrated that it is possible to reduce the number of levelling lines in a network without any lack of accuracy, if an adjustment of geometric levelling networks in all combinations plus IDW or IAHW procedure is applied. Moreover, the described algorithm in this study, can decrease the standard errors of the adjusted geopotential numbers 3-5 times in comparison to the classic variant of the adjustment. As a result, it has been shown that the accuracy of the precise geometric levelling is supreme (in the present) in comparison to other methods for determining height differences. Despite being tedious and time-consuming this method can continue to be used as a main method for establishing height systems for state territories and even continents, validation of gravimetric geoid models, monitoring the recent vertical movements of the Earth's crust, determining the difference among oceans and sees levels, predicting earthquakes, engineering tasks of the highest precision, etc.

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